A note on a theorem of Rutishauser

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1. H. Rutishauser proved the following theorem(1) :

[1] Let A be a real symmetric matrix with eigenvalues $\lambda_1>\lambda_2>\cdots>\lambda_n\geqq 0$, and with no eigenvector orthogonal to a coordinate axis. Define the sequence of matrices $A_k\,(k=0,1,2,\cdots)$ by

$$A_0=A\ ,\qquad A_k=L_kL_k' \quad\text{and}\quad A_{k+1}=L_k'L_k\ , \tag{1}$$

where L_k is a lower triangle and L_k' is the transpose of L_k. Then $\lim_{k\to\infty} A_k$ exists and is the diagonal matrix

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}.$$

In this note we prove the following slightly modified theorem :

[2] Let A be a real symmetric matrix of order n and $(Ax,x)\geqq 0$ for all real x, and let A_k be the sequence of matrices defined by (1). Then, $\lim_{k\to\infty} A_k$ exists and is a diagonal matrix whose diagonal elements are eigenvalues of A.

2. Let $A=[a_{ij}]$, $A_k=[a_{ij}^{(k)}]$ and $L_k=[l_{ij}^{(k)}]$. Then it follows from (1) that

$$a_{11}^{(k)}={l_{11}^{(k)}}^2\ ,\qquad a_{11}^{(k+1)}=\sum_{i=1}^{n}{l_{i1}^{(k)}}^2\ ,$$

$$\sum_{i=1}^{n}a_{ii}^{(k)}=tr\,(A_k)=tr\,(A_{k+1})=\sum_{i=1}^{n}a_{ii}^{(k+1)}\ .$$

From these relations we have

$$a_{11}^{(k)}\leqq a_{11}^{(k+1)}\leqq tr(A)\qquad\text{and so}\qquad \lim_{k\to\infty}a_{11}^{(k)}=d_1\ .$$

Further, $a_{11}^{(k+1)}-a_{11}^{(k)}=\sum_{i=2}^{n}{l_{i1}^{(k)}}^2\to 0\ (k\to\infty)$ implies

$$a_{1i}^{(k)}=l_{11}^{(k)}l_{i1}^{(k)}\to 0\ (k\to\infty)\quad\text{for}\quad i\geqq 2\ .$$

Let $d_p^{(k)}$ be the principal minor of order p in the upper left corner of A_k, then

$$d_p^{(k)}=(l_{11}^{(k)}l_{22}^{(k)}\cdots l_{pp}^{(k)})^2 \,.$$

By a well known theorem we have

$$d_p^{(k+1)}=\left|\begin{pmatrix} l_{11}^{(k)} & \cdots & \cdots & l_{n1}^{(k)} \\ & \ddots & & \vdots \\ 0 & & l_{pp}^{(k)} \cdots & l_{np}^{(k)} \end{pmatrix}\begin{pmatrix} l_{11}^{(k)} & & 0 \\ \vdots & \ddots & \\ \vdots & & l_{pp}^{(k)} \\ \vdots & & \vdots \\ l_{n1}^{(k)} & \cdots & l_{np}^{(k)} \end{pmatrix}\right|=\sum_{i_1,i_2,\cdots,i_p}\begin{vmatrix} l_{i_11}^{(k)} & \cdots & l_{i_p1}^{(k)} \\ \vdots & & \vdots \\ l_{i_1p}^{(k)} & \cdots & l_{i_pp}^{(k)} \end{vmatrix}^2$$

$$\geqq d_p^{(k)}+(l_{11}^{(k)}l_{22}^{(k)}\cdots l_{p-1\,p-1}^{(k)})^2\cdot\sum_{j=p+1}^{n}l_{jp}^{(k)^2} \,. \tag{2}$$

We denote by s_p the sum of all the principal minors of order p of A. Since A_k and A have the same characteristic polynomial, as is readily seen from (1), and the principal minor of a positive definite (semi-definte) matrix is positive (non-negative), we find

$$d_p^{(k)}\leqq s_p \,.$$

Hence the existence of $\lim\limits_{k\to\infty} d_p^{(k)}$.

Now let us assume A is positive definite. Then $d_p^{(k)}>0$ for all k and p. As is seen from (2),

$$\lim_{k\to\infty}(l_{11}^{(k)}l_{22}^{(k)}\cdots l_{p-1\,p-1}^{(k)})^2\left(\sum_{j=p+1}^{n}l_{jp}^{(k)^2}\right)=0$$

and

$$\lim_{k\to\infty}(l_{11}^{(k)}l_{22}^{(k)}\cdots l_{p-1\,p-1}^{(k)})^2\equiv d_{p-1}>0 \,.$$

Then, it is easy to see that

$$\lim_{k\to\infty}a_{ii}^{(k)}=\frac{d_i}{d_{i-1}} \qquad (d_0\equiv 1) \,,$$

$$\lim_{k\to\infty}a_{ij}^{(k)}=0 \quad \text{for} \quad i\neq j \,.$$

Thus the theorem [2] is proved for positive definite A.

Next we proceed to the case when A is singular. In this case $d_n^{(k)}=0$ and so $d_n=0$.

Suppose $d_p>0\,(1\leqq p\leqq m-1)$ and $d_m=0$.

In case $m=1$, since the elements of the first row and the elements of the first column of A are all 0, we can reduce the order of A by unity, and the proof of [2] follows by induction.

In case $m>1$, we can show similarly that

$$l_{11}^{(k)^2}\to d_1,\ l_{22}^{(k)^2}\to d_2/d_1,\ \cdots,\ l_{m-1m-1}^{(k)^2}\to d_{m-1}/d_{m-2}$$

and

$$l_{ij}^{(k)}\to 0 \quad \text{for} \quad i\neq j\leqq m\ .$$

Further, since $l_{jm}^{(k)}=0$ for all j and k, the elements of the m-th row or the m-th column of $A_k\,(k\geqq 1)$ are all 0. Hence by elimination of the m-th row and the m-th column of A_k, we can reduce the order of A_k by unity, and the proof of [2] follows by induction.

Reference

[1] H. Rutishauser: Une méthode pour la détermination des valeurs propres d'une matrice, C. R. Acad. Sci. Paris vol. 240 (1955) pp. 34–36.

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